Foundations of Numerical Algebraic Geometry

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Nonlinear Algebra Bootcamp
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Overview
Overview

For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^N$, solve $f(x) = 0$. 
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For a polynomial system \( f : \mathbb{C}^n \rightarrow \mathbb{C}^N \), solve \( f(x) = 0 \).

Maple

\[
> \text{solve}(x^5 - x + 1);
\]

\[
\text{RootOf}(\_Z^5 - \_Z + 1, \text{index} = 1), \text{RootOf}(\_Z^5 - \_Z + 1, \text{index} = 2), \text{RootOf}(\_Z^5 - \_Z + 1, \text{index} = 3),
\]

\[
\text{RootOf}(\_Z^5 - \_Z + 1, \text{index} = 4), \text{RootOf}(\_Z^5 - \_Z + 1, \text{index} = 5)
\]
Overview

For a polynomial system \( f : \mathbb{C}^n \rightarrow \mathbb{C}^N \), solve \( f(x) = 0 \).

Maple

```maple
> solve(x^5 - x + 1);
RootOf(_Z^5 - _Z + 1, index = 1), RootOf(_Z^5 - _Z + 1, index = 2), RootOf(_Z^5 - _Z + 1, index = 3), RootOf(_Z^5 - _Z + 1, index = 4), RootOf(_Z^5 - _Z + 1, index = 5)

> fsolve(x^5 - x + 1);
-1.167303978
```
Overview

For a polynomial system \( f : \mathbb{C}^n \to \mathbb{C}^N \), solve \( f(x) = 0 \).

Maple

```
> solve(x^5 - x + 1);
RootOf(_Z^5 - _Z + 1, index = 1), RootOf(_Z^5 - _Z + 1, index = 2), RootOf(_Z^5 - _Z + 1, index = 3),
   RootOf(_Z^5 - _Z + 1, index = 4), RootOf(_Z^5 - _Z + 1, index = 5)
```

```
> fsolve(x^5 - x + 1);
-1.167303978
```

```
> evalf(solve(x^5 - x + 1));
0.764884433600585 + 0.352471546031726 I, -0.181232444469875 + 1.08395410131771 I,
   -1.16730397826142, -0.181232444469875 - 1.08395410131771 I, 0.764884433600585
   - 0.352471546031726 I
```
Overview

For a polynomial system \( f : \mathbb{C}^n \rightarrow \mathbb{C}^N \), solve \( f(x) = 0 \).

Maple

```
> evalf(solve(x^5 - x + 1));
0.764884433600585 + 0.352471546031726 I, -0.181232444469875 + 1.08395410131771 I,
-1.16730397826142, -0.181232444469875 - 1.08395410131771 I, 0.764884433600585
- 0.352471546031726 I
```

Bertini

```
input
variable_group x;
function f;
f = x^5 - x + 1;
```

finite_solutions

```
5
7.648844336005847e-01 -3.524715460317264e-01
7.648844336005849e-01 3.524715460317262e-01
-1.812324444698754e-01 1.083954101317711e+00
-1.167303978261419e+00 -2.220446049250313e-16
-1.812324444698754e-01 -1.083954101317711e+00
```
Overview

For a polynomial system $f : \mathbb{C}^n \rightarrow \mathbb{C}^N$, solve $f(x) = 0$.

What does it mean to solve? Some examples include:
Overview

For a polynomial system \( f : \mathbb{C}^n \to \mathbb{C}^N \), solve \( f(x) = 0 \).

What does it mean to solve? Some examples include:

- Show that a solution exists
- Computing upper bounds on rank of a tensor

Strassen (1969): \( \text{rank } M_2 \leq 7 \) showing \( \omega \leq \log_2 7 < 3 \)

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\cdot
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
=
\begin{bmatrix}
  a \cdot A + b \cdot C & a \cdot B + b \cdot D \\
  c \cdot A + d \cdot C & c \cdot B + d \cdot D
\end{bmatrix}
=
\begin{bmatrix}
  I + IV - V + VII & III + V \\
  II + IV & I - II + III + VI
\end{bmatrix}
\]

\[
I : (a + d) \cdot (A + D) \\
II : (c + d) \cdot A \\
III : a \cdot (B - D) \\
IV : d \cdot (C - A) \\
V : (a + b) \cdot D \\
VI : (c - a) \cdot (A + B) \\
VII : (b - d) \cdot (C + D)
\]
Overview

For a polynomial system \( f : \mathbb{C}^n \rightarrow \mathbb{C}^N \), solve \( f(x) = 0 \).

What does it mean to solve? Some examples include:

- Compute all isolated solutions (over \( \mathbb{C} \) or \( \mathbb{R} \)).
- Number of assembly configurations

planar pentad \( SE(2) \)

spherical pentad \( SO(3) \)

Stewart-Gough platform \( SE(3) \)
Overview

For a polynomial system $f : \mathbb{C}^n \rightarrow \mathbb{C}^N$, solve $f(x) = 0$.

What does it mean to solve? Some examples include:

- Describe all irreducible components.
Overview

For a polynomial system \( f : \mathbb{C}^n \rightarrow \mathbb{C}^N \), \textbf{solve} \( f(x) = 0 \).

Generally speaking:

- Algebraic methods prefer vastly over-determined systems
  - fewer “new” polynomials to compute

- Numerical algebraic geometry prefers well-constrained systems of low degrees with coefficients of roughly unit magnitude
  - codimension = \( \# \) equations
  - stable under perturbations
Overview

For a polynomial system $f : \mathbb{C}^n \rightarrow \mathbb{C}^N$, solve $f(x) = 0$.

What does it mean to **numerically solve**?
Overview

For a polynomial system \( f : \mathbb{C}^n \to \mathbb{C}^N \), solve \( f(x) = 0 \).

What does it mean to **numerically solve**?

Need two key aspects:

- compute sufficiently accurate numerical approximation

- have an algorithm that can produce approximations of solution to any given accuracy starting from numerical approximation
  - *sufficiently accurate* depends on the algorithm
Overview

What does it mean to **numerically solve**?

- compute sufficiently accurate numerical approximation
- have an algorithm that can produce approximations of solution to any given accuracy starting from numerical approximation

Example

\[ f(x) = x^2 - 2 = 0 \]

- \( x_0 = 1 \) is *numerical solution* associated with Newton’s method
Overview

Example

\[ f(x) = x^2 - 2 = 0 \]

- \( x_0 = 1 \) is *numerical solution* associated with Newton’s method

\[ x_{k+1} = x_k - Jf(x_k)^{-1}f(x_k) \]

| \( x_0 \) | 1 |
| \( x_1 \) | 1.5 |
| \( x_2 \) | 1.41666666666666666666666666666666666667 |
| \( x_3 \) | 1.4142156862745098039215686274509803921568627450980 |
| \( x_4 \) | 1.4142135623746899106262955788901349101165596221157 |
| \( x_5 \) | 1.4142135623730950488016896235025302436149819257762 |
| \( x_6 \) | 1.4142135623730950488016887242096980785696718753772 |
| \( \sqrt{2} \) | 1.4142135623730950488016887242096980785696718753769 |
Overview

Double-edged sword of Newton’s method:

- Quadratic convergence near nonsingular solutions
- Slow convergence or divergence near singular solutions
- Difficulty away solutions (chaos, limit cycles, etc)

\[ f(x) = x^4 - 1 \]
Overview

Foundations of Numerical Algebraic Geometry:

- Continuation and path tracking
- Constructing homotopies

- Witness sets
  - Numerical Irreducible Decomposition
  - Other computations using witness sets
Continuation

Continuation from complex analysis:
- Cauchy (1789-1857), Riemann (1826-1866), Mittag-Leffler (1846-1927)
- Implicit function theorem
- Analytic extension of functions (analytic continuation)

Big picture idea:
- solutions “continue” locally under small parameter changes
Example

\[ f(x; p) = x^2 - p = 0 \]

Locally near \( p = 1 \):

\[
x(p) = \sqrt{p} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (1 - 2n)(n!)^2} (p - 1)^n
\]

- converges for \( |p - 1| \leq 1 \)
Example

\[ f(x; p) = x^2 - p = 0 \]

Locally near \( p = 1 \):

\[ x(p) = \sqrt{p} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{4^n(1 - 2n)(n!)^2} (p - 1)^n \]

▶ converges for \( |p - 1| \leq 1 \)

Use continuation to extend beyond this domain.
Example

\[ f(x; p) = x^2 - p = 0 \]

Continue the solution \( x = 1 \) at \( p = 1 \) to \( p = 1 + 2i \).
Continuation

Numerically track along the path \( x(t) \) satisfying \( H(x(t), t) = 0 \):

- (Predictor) Estimate \( x(t + \Delta t) \) from \( x(t) \) by discretizing using the Davidenko differential equation (1953):

\[
H = 0 \quad \rightarrow \quad \frac{d}{dt} H = 0 \quad \rightarrow \quad \dot{x}(t) = -J_x H(x(t), t)^{-1} J_t H(x(t), t)
\]

- Constant, Euler, Heun, Runge-Kutta, Runge-Kutta-Fehlberg, ....

- (Corrector) for each \( t \), apply Newton’s method to \( H(\bullet, t) = 0 \)
Continuation

Locally adapt both stepsize and floating-point precision:

Continuation

Certified tracking (select stepsize to guarantee to track path):


Smale’s 17th problem: polynomial time to compute a root


Pierre Lairez: 2017 SIAG/AG Early Career Prize
Example

\[ f(x; p) = x^2 - p = 0 \]

Track around a loop: \( x(e^{i\theta}) \)

\( p \in \mathbb{C} \)
Example

\[ f(x; p) = x^2 - p = 0 \]

Track around a loop: \( x(e^{i\theta}) \)

- \( \theta = 0: x = 1 \)
- \( \theta = 2\pi: x = -1 \)
- \( \theta = 4\pi: x = 1 \)

\( p \in \mathbb{C} \)

\[ \text{real}(x) \quad \text{imag}(x) \]

\text{cycle number = winding number = 2}
Continuation

\[ f(x; p) = x^2 - p = 0 \]

Track around a loop: \( x(e^{i\theta}) \)

- monodromy action: permutation of solutions along loop
  - compute other solutions
  - decompose solution sets

real(\( x \))  imag(\( x \))
Example

\[ f(x; p) = x^2 - p = 0 \]

Track around a loop: \( x(e^{i\theta}) \)

- Cauchy integral theorem: computing singular endpoints
  - cycle number \( c \)
  - sufficiently small radius \( r > 0 \)

\[
x(0) = \frac{1}{2\pi c} \int_0^{2\pi c} x(re^{i\theta}) d\theta
\]

Isolated Solutions

Find all isolated solutions of

\[ f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0 \]
Isolated Solutions

\[ f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0 \]

Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to "continue"
   - think of \( f \) as a member of a family \( \mathcal{F} \)
Isolated Solutions

Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to “continue”
   - think of $f$ as a member of a family $\mathcal{F}$

2. homotopy that describes the deformation of the parameters
   - construct a deformation inside of $\mathcal{F}$ that ends at $f$
Isolated Solutions

Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to “continue”
   - think of $f$ as a member of a family $\mathcal{F}$

2. homotopy that describes the deformation of the parameters
   - construct a deformation inside of $\mathcal{F}$ that ends at $f$

3. start points to track along paths as parameters deform
   - parallelize computation – track each path independently
Isolated Solutions

Theorem

For properly constructed homotopies, with finite endpoints $S \subset \mathbb{C}^n$:

- each isolated solution is contained in $S$
  - in fact, $S$ contains a point on every connected component
- for square systems, multiplicity = number of paths if isolated.
Isolated Solutions

Art in the construction of family $\mathcal{F}$:

- number of start points
- ease to compute start points

Each method is sharp for generic members of $\mathcal{F}$. 
Example

Isolated Solutions

\[ f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix} \]
Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Bézout family (total degree):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x, y) \\ g_2(x, y) \end{bmatrix} : \deg g_i = 2 \right\} \quad g = \begin{bmatrix} x^2 - 1 \\ y^2 - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for $g$: 4

$$H = (1 - t) \cdot f + \gamma t \cdot g$$

- $\gamma \in \mathbb{C}$ is used to create a general deformation
  - avoid singularities that arise from tracking over real numbers
Isolated Solutions

\[ f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix} \]

- Bézout family (total degree):

\[ F = \left\{ \left[ \begin{array}{c} g_1(x, y) \\ g_2(x, y) \end{array} \right] : \deg g_i = 2 \right\} \quad g = \begin{bmatrix} x^2 - 1 \\ y^2 - 1 \end{bmatrix} \]

Number of paths = number of isolated solutions for \( g \): 4

Bertini

```plaintext
finite_solutions

1

2.000000000000000e+00 0.000000000000000e+00
-1.666666666666667e-01 0.000000000000000e+00
```

input

```plaintext
variable_group x, y;
function f1, f2;
f1 = x^2 + 2*x - 8;
f2 = x*y + 2*x + 4*y - 3;
```
Isolated Solutions

$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$

- Multihomogeneous Bézout family (Morgan-Sommese (1987)):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x) \\ g_2(x, y) \end{bmatrix} : \begin{array}{l} \deg_x g_1 = 2, \\ \deg_x g_2 = \deg_y g_2 = 1 \end{array} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 2)(y - 1) \end{bmatrix} \quad \text{H} = (1 - t) \cdot f + \gamma t \cdot g$$

Number of paths = number of isolated solutions for $g$: 2
Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

* Multihomogeneous Bézout family (Morgan-Sommese (1987)):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x) \\ g_2(x, y) \end{bmatrix} : \deg_x g_1 = 2, \quad \deg_x g_2 = \deg_y g_2 = 1 \right\}$$

Number of paths = number of isolated solutions for $g$: 2

Bertini

```plaintext
input variable_group x;
variable_group y;
function f1,f2;
f1 = x^2 + 2*x - 8;
f2 = x*y + 2*x + 4*y - 3;
```
Example

\[ f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix} \]

- Polyhedral (BKK, Huber-Sturmfels (1995)):

\[ \mathcal{F} = \left\{ \begin{bmatrix} a_1 x^2 + a_2 x + a_3 \\ a_4 xy + a_5 x + a_6 y + a_7 \end{bmatrix} : a_i \in \mathbb{C} \right\} \]

\[ g = \begin{bmatrix} x^2 - 1 \\ y - 1 \end{bmatrix} \]

\[ H = (1 - t) \cdot f + \gamma t \cdot g \]

Number of paths = number of isolated solutions for \( g \): 2
Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Extra structure in the coefficients of $f$.

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1a_2 \\ (x - a_1)y + a_3x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 1)y - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for $g$: 1
Example

\[ f = \begin{bmatrix}
  x^2 + 2x - 8 \\
  xy + 2x + 4y - 3
\end{bmatrix} \]

\[ \mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix}
  x^2 - (a_1 + a_2)x + a_1a_2 \\
  (x - a_1)y + a_3x + a_4
\end{bmatrix} : a_i \in \mathbb{C} \right\} \]

\[ g = \begin{bmatrix}
  x^2 - 1 \\
  (x - 1)y - 1
\end{bmatrix} \]

Since \( \mathcal{F} \) is no longer linear, use a parameter homotopy:

\[ H = p(x, y; a(t)) \]

where \[ a(t) = (1 - \tau(t))(-4, 2, 2, -3) + \tau(t)(1, -1, 0, -1) \]

\[ \tau(t) = \frac{\gamma t}{1 - t + \gamma t} \]
Example

\[ f = \begin{bmatrix}
  x^2 + 2x - 8 \\
  xy + 2x + 4y - 3
\end{bmatrix} \]

\[ \mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix}
  x^2 - (a_1 + a_2)x + a_1a_2 \\
  (x - a_1)y + a_3x + a_4
\end{bmatrix} : a_i \in \mathbb{C} \right\} \]
Isolated Solutions

Some software options:

- Bertini
- Bertini.m2
- Hom4PS
- HomotopyContinuation.jl
- MonodromySolver
- NAG4M2
- Paramotopy
- PHCpack

Visitors to ICERM: Bates, Brake, Chen, Duff, Hill, Lee, Leykin, Rodriguez, Sommars, Sommese, Wampler, ...
Isolated Solutions

Example (Alt’s problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.
Isolated Solutions

Example (Alt’s problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.

\[ 8652 = 6 \cdot 1442 \quad \text{(Wampler-Morgan-Sommese (1992))} \]

Their polynomial system: 4 quadratics and 8 quartics

<table>
<thead>
<tr>
<th></th>
<th>Bézout</th>
<th>$1,048,576 = 2^4 \cdot 4^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-hom Bézout</td>
<td>286,720</td>
<td>$2^{12} \cdot \binom{8}{4}$</td>
</tr>
<tr>
<td>Polyhedral</td>
<td>79,135</td>
<td></td>
</tr>
<tr>
<td>Product decomp.</td>
<td>18,700</td>
<td></td>
</tr>
<tr>
<td>Actual</td>
<td>8,652</td>
<td></td>
</tr>
</tbody>
</table>
Isolated Solutions

Mechanical Design 101 | MECHANICAL DESIGN EDUCATIONAL RESOURCE

US Patents: four-bar, six-bar, eight-bar linkages

<table>
<thead>
<tr>
<th>Period</th>
<th>Four-bar</th>
<th>Six-bar</th>
<th>Eight-bar</th>
</tr>
</thead>
<tbody>
<tr>
<td>1976-1980</td>
<td>194</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1981-1985</td>
<td>231</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>1986-1990</td>
<td>230</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>1991-1995</td>
<td>296</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>1996-2000</td>
<td>431</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>2001-2005</td>
<td>543</td>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>2006-2010</td>
<td>705</td>
<td>19</td>
<td>4</td>
</tr>
<tr>
<td>2011-2015</td>
<td>989</td>
<td>24</td>
<td>4</td>
</tr>
</tbody>
</table>

Number of Patents

Five year periods from 1976 to 2015
Witness Set

Describe all solutions of

\[ f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_k(x_1, \ldots, x_n) \end{bmatrix} = 0 \]
Witness Set

How to represent an irreducible algebraic variety $A$ on a computer?
Witness Set

How to represent an irreducible algebraic variety \( A \) on a computer?

- algebraic: prime ideal \( I(A) = \{g \mid g(a) = 0 \text{ for all } a \in A\} \)
  - Hilbert Basis Theorem (1890): there exists \( f_1, \ldots, f_k \) such that
    \[
    I(A) = \langle f_1, \ldots, f_k \rangle
    \]
How to represent an irreducible algebraic variety $A$ on a computer?

- geometric: witness set $\{f, \mathcal{L}, \mathcal{W}\}$ where
  - $f$ is polynomial system where $A$ is an irreducible component of $\mathcal{V}(f)$
  - $\mathcal{L}$ is a linear space with $\text{codim} \mathcal{L} = \dim A$
  - $\mathcal{W} = \mathcal{L} \cap A$ where $\# \mathcal{W} = \deg A$

- Witness sets “localize” computations to $A$ effectively ignoring other irreducible components
- Sample points from $A$ by moving the linear slice $\mathcal{L}$
Example

\[ A = \{ [s^3, s^2t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1 \} \subset \mathbb{P}^3 \] - twisted cubic curve
Example

\[ A = \{[s^3, s^2 t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1\} \subset \mathbb{P}^3 \] - twisted cubic curve

\[ \{f, \mathcal{L}, W\} \]

- \[ f = \begin{bmatrix} x_1^2 - x_0 x_2 \\ x_1 x_2 - x_0 x_3 \end{bmatrix} \]

- \[ \mathcal{L} = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid 6x_0 - 6x_1 - 2x_2 + x_3 = 0\} \subset \mathbb{P}^3 \]
  - \[ \text{codim } \mathcal{L} = \text{dim } A = 1 \]

- \[ W = \left\{ \begin{array}{c} [1, 3.2731, 10.7130, 35.0644], \\ [1, 0.8596, 0.7389, 0.6351], \\ [1, -2.1326, 4.5481, -9.6995] \end{array} \right\} \]
  - \[ \text{deg } A = 3 \]
Witness Set

Numerical irreducible decomposition:

- compute a witness set for each irreducible component
Example

\[ f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix} \]

Witness Set

Bertini input

CONFIG
TrackType: 1;
END;
INPUT
hom_variable_group x0,x1,x2,x3;
function f1,f2;
f1 = x1^2 - x0*x2;
f2 = x1*x2 - x0*x3;
END;

Dimension 1: 2 classified components

degree 1: 1 component
degree 3: 1 component
Witness Set

Reduce to codimension $= \# \text{ equations}$ via randomization:

Theorem (Bertini)

Let $f : \mathbb{C}^n \to \mathbb{C}^N$ and $A \subset V(f) \subset \mathbb{C}^n$ be an irreducible component with codim $A = c$. If $R \in \mathbb{C}^{c \times N}$ is general, then

- $A$ is an irreducible component of $V(R \cdot f)$
- $V(R \cdot f) \setminus V(f)$ is either empty or smooth of codimension $c$. 
Witness Set

Reduce to codimension $= \# \text{ equations}$ via randomization:

Theorem (Bertini)

Let $f : \mathbb{C}^n \to \mathbb{C}^N$ and $A \subset V(f) \subset \mathbb{C}^n$ be an irreducible component with codim $A = c$. If $R \in \mathbb{C}^{c \times N}$ is general, then

- $A$ is an irreducible component of $V(R \cdot f)$
- $V(R \cdot f) \setminus V(f)$ is either empty or smooth of codimension $c$.

Example

For general $R \in \mathbb{C}^{2 \times 3}$ and $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \end{bmatrix}$,

- $V(R \cdot f) = \text{twisted cubic} + \text{line}$
Example

\[
f = \begin{bmatrix}
(x - y)(\hat{x} - \hat{y}) \\
(x - y)(\hat{a}\hat{x} - 2\hat{a}\hat{y} + 2\hat{b}\hat{x} - \hat{b}\hat{y}) \\
(\hat{x} - \hat{y})(a\hat{x} - 2a\hat{y} + 2b\hat{x} - by) \\
\hat{a}\hat{b}(x - y)(\hat{a}\hat{y} - \hat{b}\hat{x}) \\
ab(\hat{x} - \hat{y})(ay - bx) \\
\vdots \\
(ab\hat{x}\hat{y} - \hat{a}bx\hat{y})(ab\hat{x} - \hat{a}bx - ab\hat{y} + \hat{a}by - a\hat{x}\hat{y} + \hat{a}x\hat{y} + b\hat{x}\hat{y} - \hat{b}\hat{x}\hat{y}) \\
(ab\hat{x}\hat{y} - \hat{a}bx\hat{y})(ab\hat{x} - \hat{a}bx - ab\hat{y} + \hat{a}by - a\hat{x}\hat{y} + \hat{a}x\hat{y} + b\hat{x}\hat{y} - \hat{b}\hat{x}\hat{y})
\end{bmatrix}
\]

15 polynomials in 8 variables \(a, b, x, y, \hat{a}, \hat{b}, \hat{x}, \hat{y}\)

For general \(R \in \mathbb{C}^{8 \times 15}\):

- \(V(R \cdot f) \setminus V(f)\) consists of finitely many points
  - all nonsingular with respect to \(R \cdot f = 0\)
Witness Set

Example

\[
f = \begin{bmatrix}
(x - y)(\hat{x} - \hat{y}) \\
(x - y)(\hat{a}\hat{x} - 2\hat{a}\hat{y} + 2\hat{b}\hat{x} - \hat{b}\hat{y}) \\
(\hat{x} - \hat{y})(\hat{a}x - 2\hat{a}y + 2\hat{b}x - \hat{b}y) \\
\hat{a}\hat{b}(x - y)(\hat{a}\hat{y} - \hat{b}\hat{x}) \\
ab(\hat{x} - \hat{y})(ay - bx) \\
\vdots \\
(ab\hat{x}y - \hat{a}bx\hat{y})(ab\hat{x} - \hat{a}bx - \hat{a}by + \hat{a}by - \hat{a}\hat{x}\hat{y} + \hat{a}\hat{y} + \hat{b}\hat{x}\hat{y} - \hat{b}\hat{y}) \\
(ab\hat{x}y - \hat{a}bx\hat{y})(ab\hat{x} - \hat{a}bx - \hat{a}by + \hat{a}by - \hat{a}\hat{x}\hat{y} + \hat{a}\hat{y} + \hat{b}\hat{x}\hat{y} - \hat{b}\hat{y})
\end{bmatrix}
\]

15 polynomials in 8 variables \(a, b, x, y, \hat{a}, \hat{b}, \hat{x}, \hat{y}\)

For general \(R \in \mathbb{C}^{8 \times 15}\):
- \(V(R \cdot f) \setminus V(f)\) consists of finitely many points
  - all nonsingular with respect to \(R \cdot f = 0\)
- Using Bertini: \(|V(R \cdot f) \setminus V(f)| = 8652\)
  - Proving this would complete proof of Alt’s problem
Witness Set

Given $\mathcal{W} \subseteq V(f) \cap \mathcal{L}$, how to test that $\mathcal{W} = \mathcal{L} \cap A$ for some variety $A \subseteq V(f)$?

- Trace test: centroid moves linearly as slices move in parallel.
Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- membership testing: is $x^* \in A$?
  - decide if $g(x^*) = 0$ for every $g \in I(A)$ without knowing $I(A)$
Witness Set

- projection: $\overline{\pi(A)}$
- perform computations on $\overline{\pi(A)}$ without knowing any polynomials that vanish on $\overline{\pi(A)}$
Witness Set

- intersection: $A \cap B$
  - special case is regeneration
    - $\mathcal{V}(f_1, \ldots, f_k, f_{k+1}) = \mathcal{V}(f_1, \ldots, f_k) \cap \mathcal{V}(f_{k+1})$ via witness sets
  - compute $A_{\text{sing}}$
  - compute critical points of optimization problem

\[ \min ||x^* - a||_2 \quad \text{such that} \quad a \in A \cap \mathbb{R}^n \]
Witness Set

Test other algebraic properties of $A$

- is $A$ arithmetically Cohen Macaulay?
- is $A$ arithmetically Gorenstein?
- is $A$ a complete intersection?
Summary

Numerical algebraic geometry provides a toolbox for solving polynomial systems.

- “If a problem was easy, someone else would have solved it.”
  - Gröbner basis computation probably did not terminate

- Think carefully about what information you want/need

- Art in building efficient homotopies that incorporate structure

- Preconditioning is important
  - Transform problem into form suitable for numerical computations
Thank You!